REPRESENTATIONS OF CERTAIN BANACH ALGEBRAS

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Abstract. We prove that certain Banach subalgebras H of $C^*(X)$ are isometrically isomorphic to $C_\infty(Y)$, for some unique (up to homeomorphism) locally compact Hausdorff space Y. The space Y is explicitly constructed as a subspace of the Stone–Čech compactification of X. The known construction of Y enables us to examine certain properties of either H or Y and derive results not expected to be deducible from the standard Gelfand Theory.

1. Introduction

By a *space* we mean a *topological space*; completely regular spaces are Hausdorff. Throughout this note the underlying field of scalars (which is fixed throughout each discussion) is assumed to be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, unless specifically stated otherwise.

Let X be a completely regular space. Denote by $C^*(X)$ the set of all continuous bounded scalar-valued functions on X. If $f \in C^*(X)$, the *zero-set* of f, denoted by $\mathrm{Z}(f)$, is $f^{-1}(0)$, the *cozero-set* of f, denoted by $\mathrm{Coz}(f)$, is $X\backslash\mathrm{Z}(f)$, and the *support* of f, denoted by $\mathrm{supp}(f)$, is $\mathrm{cl}_X\mathrm{Coz}(f)$. Let

$$\mathrm{Coz}(X) = \big\{\mathrm{Coz}(f) : f \in C^*(X)\big\}.$$

The elements of $\mathrm{Coz}(X)$ are called *cozero-sets* of X. Denote by $C_\infty(X)$ the set of all $f \in C^*(X)$ which vanish at infinity (i.e., $|f|^{-1}([\epsilon,\infty))$ is compact for each $\epsilon > 0$) and denote by $C_K(X)$ the set of all $f \in C^*(X)$ with compact support.

This work is a continuation of our previous work [9] (and [10], though it is self contained) in which we have studied the Banach algebra of continuous bounded scalar-valued functions with separable support defined on a locally separable metrizable space X. Indeed, this article is an outgrowth of the author's unsuccessful (partly successful, however) attempt to give an alternative proof of the celebrated commutative Gelfand–Naimark Theorem. We show that certain Banach subalgebras H of $C^*(X)$ are representable as $C_\infty(Y)$ for some unique locally compact Hausdorff space Y. We construct Y explicitly as a subspace of the Stone–Čech compactification of X. The known construction of Y enables us to examine certain properties of either H or Y and derive results not expected to be deducible from the standard Gelfand Theory.

We now briefly review certain facts from General Topology. Additional information may be found in [3] and [4].

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1.1. The Stone–Čech compactification. Let X be a completely regular space. By a *compactification* γX of X we mean a compact Hausdorff space γX containing X as a dense subspace. The *Stone–Čech compactification* βX of X is the compactification of X which is characterized among all compactifications of X by the following property: Every continuous $f : X \to K$, where K is a compact Hausdorff space, is continuously extendable over βX; denote by f_β this continuous extension of f. The Stone–Čech compactification always exists. Use will be made in what follows of the following properties of βX.

- X is locally compact if and only if X is open in βX.
- Any open-closed subspace of X has open-closed closure in βX.
- If $X \subseteq T \subseteq \beta X$ then $\beta T = \beta X$.
- If X is normal then $\beta T = \mathrm{cl}_{\beta X} T$ for any closed subspace T of X.

1.2. Locally-$\mathscr{P}$ spaces. Let $\mathscr{P}$ be a topological property. A space X is called *locally-$\mathscr{P}$*, if each $x \in X$ has an open neighborhood U in X whose closure $\mathrm{cl}_X U$ has $\mathscr{P}$.

1.3. Metrizable spaces and separability. The *density* of a space X, denoted by $d(X)$, is the smallest cardinal number of the form $|D|$, where D is dense in X. Therefore, a space X is separable if $d(X) \leq \aleph_0$. Note that in any metrizable space the three notions of separability, being Lindelöf, and second countability coincide; thus any subspace of a separable metrizable space is separable. By a theorem of Alexandroff, any locally separable metrizable space X can be represented as a disjoint union $X = \bigcup_{i \in I} X_i$, where I is an index set, and X_i is a non-empty separable open-closed subspace of X for each $i \in I$. (See Problem 4.4.F of [3].) Note that $d(X) = |I|$ if I is infinite.

1.4. Paracompact spaces and the Lindelöf property. Let X a regular space. For any open covers $\mathscr{U}$ and $\mathscr{V}$ of X we say that $\mathscr{U}$ is a *refinement* of $\mathscr{V}$ if every element of $\mathscr{U}$ is contained in an element of $\mathscr{V}$. An open cover $\mathscr{U}$ of X is called *locally finite* if each point of X has an open neighborhood in X intersecting only a finite number of the elements of $\mathscr{U}$. The space X is called *paracompact* if for every open cover $\mathscr{U}$ of X there is an open cover of X which refines $\mathscr{U}$. Every metrizable space is paracompact and every paracompact space is normal. The *Lindelöf number* of X, denoted by $\ell(X)$, is the smallest cardinal number $\mathfrak{m}$ such that every open cover of X has a subcover of cardinality $\leq \mathfrak{m}$. Therefore, X is Lindelöf if $\ell(X) \leq \aleph_0$. Any locally compact paracompact space X can be represented as a disjoint union $X = \bigcup_{i \in I} X_i$, where I is an index set, and X_i is a non-empty Lindelöf open-closed subspace of X for each $i \in I$. (See Theorem 5.1.27 of [3].) Note that $\ell(X) = |I|$ if I is infinite.

2. The representation theorem

The subspace $\lambda_H X$ of βX defined below plays a crucial role in our study.

Definition 2.1. Let X be a completely regular space and let $H \subseteq C^*(X)$. Define

$$\lambda_H X = \bigcup \big\{ \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} \mathrm{Coz}(h) : h \in H \big\}.$$

The above definition of $\lambda_H X$ is motivated by the definition of $\lambda_{\mathscr{P}} X$ (here $\mathscr{P}$ is a topological property) as given in [6] (also, in [7] and [8]). Note that $\lambda_H X$ is open in βX and is thus locally compact.

A version of the classical Banach–Stone Theorem states that if X and Y are locally compact Hausdorff spaces, the Banach algebras $C_\infty(X)$ and $C_\infty(Y)$ are isometrically isomorphic if and only if the spaces X and Y are homeomorphic (see Theorem 7.1 of [2]); this will be used in the proof of the next theorem.

Recall that if X is a space and D is a dense subspace of X, then

$$\mathrm{cl}_X U = \mathrm{cl}_X(U \cap D)$$

for any open subspace U of X.

Theorem 2.2. *Let X be a completely regular space. Let H be a Banach subalgebra of $C^*(X)$ such that*

(1) *For any $x \in X$ there is some $h \in H$ with $h(x) \neq 0$.*
(2) *For any $f \in C^*(X)$, if $\mathrm{supp}(f) \subseteq \mathrm{supp}(h)$ for some $h \in H$, then $f \in H$.*

Then H is isometrically isomorphic to $C_\infty(Y)$ for some unique locally compact Hausdorff space Y, namely $Y = \lambda_H X$.

Proof. We divide the proof into verification of several claims.

Claim. $X \subseteq \lambda_H X$.

Proof of the claim. Let $x \in X$. By (1) there is some $h \in H$ with $h(x) \neq 0$. Since

$$\mathrm{Coz}(h_\beta) \subseteq \mathrm{cl}_{\beta X}\mathrm{Coz}(h_\beta) = \mathrm{cl}_{\beta X}\big(X \cap \mathrm{Coz}(h_\beta)\big) = \mathrm{cl}_{\beta X}\mathrm{Coz}(h)$$

we have

$$x \in \mathrm{Coz}(h_\beta) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}\mathrm{Coz}(h) \subseteq \lambda_H X.$$

This proves the claim.

For any $f \in C^*(X)$ denote $f_H = f_\beta|\lambda_H X$. By the above claim f_H extends f. Note that $\mathrm{supp}(|f|) = \mathrm{supp}(f)$ for any $f \in C^*(X)$; thus by (2) we have $|f| \in H$ if $f \in H$.

Claim. *For any $f \in C^*(X)$ the following are equivalent:*

(a) *$f \in H$.*
(b) *$f_H \in C_\infty(\lambda_H X)$.*

Proof of the claim. (a) *implies* (b). Arguing as in the proof of the first claim we have $\mathrm{Coz}(h_\beta) \subseteq \lambda_H X$. If $\epsilon > 0$ then

$$|f_H|^{-1}\big([\epsilon, \infty)\big) = |f_\beta|^{-1}\big([\epsilon, \infty)\big)$$

is closed in βX and is therefore compact.

(b) *implies* (a). Let n be a positive integer. Since $|f_H|^{-1}([1/n, \infty))$ is a compact subspace of $\lambda_H X$, we have

$$\begin{aligned}
|f_H|^{-1}\big([1/n, \infty)\big) &\subseteq \bigcup_{i=1}^{k_n} \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}\mathrm{Coz}(h_i) \\
&\subseteq \bigcup_{i=1}^{k_n} \mathrm{cl}_{\beta X}\mathrm{Coz}(h_i) \\
&= \mathrm{cl}_{\beta X}\Big(\bigcup_{i=1}^{k_n} \mathrm{Coz}(h_i)\Big) \\
&= \mathrm{cl}_{\beta X}\Big(\bigcup_{i=1}^{k_n} \mathrm{Coz}\big(|h_i|\big)\Big) = \mathrm{cl}_{\beta X}\mathrm{Coz}\Big(\sum_{i=1}^{k_n} |h_i|\Big)
\end{aligned}$$

for some $h_1, \dots, h_{k_n} \in H$. Let

$$g_n = |h_1| + \cdots + |h_{k_n}| \in H.$$

We have

$$\begin{aligned} |f|^{-1}\big([1/n, \infty)\big) &= X \cap |f_H|^{-1}\big([1/n, \infty)\big) \\ &\subseteq X \cap \mathrm{cl}_{\beta X}\mathrm{Coz}(g_n) = \mathrm{cl}_X \mathrm{Coz}(g_n) = \mathrm{supp}(g_n). \end{aligned}$$

Let

$$g = \sum_{n=1}^{\infty} 2^{-n} \frac{g_n}{\|g_n\|}.$$

(We may assume that $g_n \neq 0$ for each positive integer n.) Then $g \in H$. Since

$$\mathrm{Coz}(f) = \bigcup_{n=1}^{\infty} |f|^{-1}\big([1/n, \infty)\big) \subseteq \bigcup_{n=1}^{\infty} \mathrm{supp}(g_n) \subseteq \mathrm{supp}(g)$$

we have $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$, which by (2) yields $f \in H$. This proves the claim.

Claim. *Let $\psi : H \to C_\infty(\lambda_H X)$ be defined by $\psi(h) = h_H$ for any $h \in H$. Then ψ is an isometric isomorphism.*

Proof of the claim. By the second claim the function ψ is well-defined. It is clear that ψ is a homomorphism and that ψ is injective. (Note that $X \subseteq \lambda_H X$ by the first claim and that any two scalar-valued continuous functions on $\lambda_H X$ coincide, provided that they agree on X.) To show that ψ is surjective, let $g \in C_\infty(\lambda_H X)$. Then $(g|X)_H = g$ and thus $g|X \in H$ by the second claim. Now $\psi(g|X) = g$. Finally, to show that ψ is an isometry, let $h \in H$. Then

$$|h_H|(\lambda_H X) = |h_H|(\mathrm{cl}_{\lambda_H X} X) \subseteq \mathrm{cl}_{\mathbb{R}}\big(|h_H|(X)\big) = \mathrm{cl}_{\mathbb{R}}\big(|h|(X)\big) \subseteq \big[0, \|h\|\big]$$

which yields $\|h_H\| \leq \|h\|$. That $\|h\| \leq \|h_H\|$ is clear, as h_H extends h. This proves the claim.

The uniqueness part of the theorem follows from the Banach–Stone Theorem. □

The following corollary of Theorem 2.2 also is of interest.

Theorem 2.3. *Let X be a completely regular space. Let H be a Banach subalgebra of $C^*(X)$ such that*

($\star$) *For any $x \in X$ there is some $h \in H$ with $h(x) \neq 0$.*

Then H may be isometrically imbedded into $C_\infty(Y)$ for some locally compact Hausdorff space Y, namely $Y = \lambda_H X$.

Proof. Let

$$G = \big\{f \in C^*(X) : \mathrm{supp}(f) \subseteq \mathrm{supp}(h) \text{ for some } h \in H\big\}.$$

We show that G is a Banach subalgebra of $C^*(X)$ satisfying conditions (1)–(2) of Theorem 2.2. By Theorem 2.2 it then follows that G is isometrically isomorphic to $C_\infty(Y)$ where $Y = \lambda_G X$. (Note that Y is a locally compact Hausdorff space.) We then observe that $\lambda_G X = \lambda_H X$. Since G contains H, this will complete the proof.

To show that G is a subalgebra of $C^*(X)$, let $g_i \in G$, where $i = 1, 2$, and let $h_i \in H$ be such that $\mathrm{supp}(g_i) \subseteq \mathrm{supp}(h_i)$. Then

$$\mathrm{supp}(g_1 + g_2) \subseteq \mathrm{supp}(g_1) \cup \mathrm{supp}(g_2) \subseteq \mathrm{supp}(h_1) \cup \mathrm{supp}(h_2) \subseteq \mathrm{supp}(h_1^2 + h_2^2).$$

Since $h_1^2 + h_2^2 \in H$ we have $g_1 + g_2 \in G$. Also, $g_1 g_2 \in G$, as $\operatorname{supp}(g_1 g_2) \subseteq \operatorname{supp}(g_1)$ (and $\operatorname{supp}(g_1) \subseteq \operatorname{supp}(h_1)$). Similarly, $cg \in G$ if $g \in G$ and c is a scalar. Therefore, G is a subalgebra of $C^*(X)$.

To show that G is a Banach space we show that G is closed in $C^*(X)$. Let $g_1, g_2, \ldots$ be a sequence in G converging to some $f \in C^*(X)$. For each positive integer n let $h_n \in H$ be such that $\operatorname{supp}(g_n) \subseteq \operatorname{supp}(h_n)$. (We may assume that $h_n \neq 0$.) Since

$$\operatorname{Coz}(f) \subseteq \bigcup_{n=1}^{\infty} \operatorname{Coz}(g_n) \subseteq \bigcup_{n=1}^{\infty} \operatorname{supp}(g_n) \subseteq \bigcup_{n=1}^{\infty} \operatorname{supp}(h_n) \subseteq \operatorname{supp}(h)$$

where

$$h = \sum_{n=1}^{\infty} 2^{-n} \frac{h_n^2}{\|h_n^2\|}$$

we have $\operatorname{supp}(f) \subseteq \operatorname{supp}(h)$. Since $h \in H$ it then follows that $f \in G$.

Note that G satisfies condition (1) of Theorem 2.2, as (by the definition of G) G contains H and (by ($\star$)) H does. That G satisfies condition (2) of Theorem 2.2 is obvious and follows from the definition of G.

To conclude the proof we need to show that $\lambda_G X = \lambda_H X$. That $\lambda_H X \subseteq \lambda_G X$ is obvious, as $H \subseteq G$. The reverse inclusion $\lambda_G X \subseteq \lambda_H X$ also is obvious and follows from the definition of G (and of $\lambda_G X$ and $\lambda_H X$). $\square$

3. Examples

In this section we give examples of spaces X and Banach subalgebras H of $C^*(X)$ for which Theorem 2.2 is applicable.

Recall that a topological property $\mathscr{P}$ is called *hereditary with respect to closed subspaces*, if each closed subspace of a space with $\mathscr{P}$ has $\mathscr{P}$.

Theorem 3.1. *Let $\mathscr{Q}$ be a topological property hereditary with respect to closed subspaces. Let $\mathscr{P}$ be a topological property such that*

1. *$\emptyset$ has $\mathscr{P}$.*
2. *$\mathscr{P}$ is hereditary with respect to closed subspaces of spaces with $\mathscr{Q}$.*
3. *Any space with $\mathscr{Q}$ containing a dense subspace with $\mathscr{P}$ has $\mathscr{P}$.*
4. *Any space representable as the union of a countable number of its closed subspaces each with $\mathscr{P}$ has $\mathscr{P}$.*

Let X be a completely regular locally-$\mathscr{P}$ space with $\mathscr{Q}$. Let

$$H = \big\{ f \in C^*(X) : \operatorname{supp}(f) \text{ has } \mathscr{P} \big\}.$$

Then H is a Banach algebra isometrically isomorphic to $C_\infty(Y)$ for some unique locally compact Hausdorff space Y, namely $Y = \lambda_H X$.

Proof. We verify that H satisfies the assumption of Theorem 2.2. First, we need to show that H is a subalgebra of $C^*(X)$. Note that $0 \in H$ by (1). Let $f, g \in H$. Then $\operatorname{supp}(f) \cup \operatorname{supp}(g)$ has $\mathscr{Q}$, as it is closed in X, and has $\mathscr{P}$, as it is the union of two of its closed subspaces each with $\mathscr{P}$; see (4). Since

$$\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$$

it then follows that $\operatorname{supp}(f+g)$ has $\mathscr{P}$, by (2). That is $f + g \in H$. Analogously, $fg \in H$ and $cf \in H$ for any scalar c. (Note that $\operatorname{supp}(fg) \subseteq \operatorname{supp}(f)$.)

Next, we show that H is closed in $C^*(X)$. Let $h_1, h_2, \dots$ be a sequence in H converging to some $f \in C^*(X)$. Note that

$$E = \mathrm{cl}_X\Big(\bigcup_{n=1}^{\infty} \mathrm{supp}(h_n)\Big)$$

has $\mathscr{Q}$, as it is closed in X, and thus by (3) has $\mathscr{P}$, as it contains $\bigcup_{n=1}^{\infty} \mathrm{supp}(h_n)$ as a dense subspace, and by (4) the latter has $\mathscr{P}$, as it is a countable union of its closed subspaces each with $\mathscr{P}$. Note that

$$\mathrm{Coz}(f) \subseteq \bigcup_{n=1}^{\infty} \mathrm{Coz}(h_n).$$

Therefore $\mathrm{supp}(f)$ has $\mathscr{P}$ by (2), as it is closed in E. That is $f \in H$.

This shows that H is a Banach subalgebra of $C^*(X)$. Next, we verify that H satisfies conditions (1) and (2) of Theorem 2.2.

To show that H satisfies condition (1) of Theorem 2.2, let $x \in X$. Let U be an open neighborhood of x in X such that $\mathrm{cl}_X U$ has $\mathscr{P}$. Note that $\mathrm{cl}_X U$ has also $\mathscr{Q}$, as it is closed in X. Let $f : X \to [0,1]$ be continuous with $f(x) = 1$ and $f|(X \backslash U) \equiv 0$. Then $\mathrm{supp}(f)$ has $\mathscr{P}$ by (2), as it is closed in $\mathrm{cl}_X U$. Therefore $f \in H$.

That H satisfies condition (2) of Theorem 2.2 is clear, as if $f \in C^*(X)$ with $\mathrm{supp}(f) \subseteq \mathrm{supp}(h)$ for some $h \in H$, then $\mathrm{supp}(h)$ has $\mathscr{P}$ (and also $\mathscr{Q}$, as it is closed in X) and thus, by (2) again, so does its closed subspace $\mathrm{supp}(f)$. Therefore $f \in H$. □

Remark 3.2. Observe that in the proof of Theorem 3.1 we actually proved that H is a closed subalgebra of $C^*(X)$.

Remark 3.3. The set

$$C_{\mathscr{P}}(X) = \big\{f \in C^*(X) : \mathrm{supp}(f) \text{ has } \mathscr{P}\big\}$$

where X is a space and $\mathscr{P}$ is a topological property, has been also considered in [1] and [11]. In [1] (Theorem 2.2) conditions are given which are necessary and sufficient for $C_{\mathscr{P}}(X)$ to be a Banach space. The approach in [11] is quite algebraic.

The next two theorems are to provide examples of topological properties $\mathscr{P}$ and $\mathscr{Q}$ satisfying the assumption of Theorem 3.1. Theorem 3.4 (1)–(4) is known (see [9]); we sketch the proof here for its application in the proof of Theorem 3.5 (and completeness of results). We will use the following facts.

A space X is called *countably compact* if every countable open cover of X has a finite subcover, equivalently, if every countable infinite subset of X has a limit point in X. If X is a locally compact Hausdorff space then $C_\infty(X) = C_K(X)$ if and only if every σ-compact subspace of X is contained in a compact subspace of X (see Problem 7G.2 of [4]); in particular, $C_\infty(X) = C_K(X)$ implies that X is countably compact.

Let D be an uncountable discrete space. Denote by D_λ the subspace of βD consisting of elements in the closure in D of countable subsets of D. In [12], the author proves the existence of a continuous (2-valued) function $f : D_\lambda \backslash D \to [0,1]$ which is not continuously extendible over $\beta D \backslash D$. This, in particular, proves that D_λ is not normal. (To see this, suppose the contrary. Note that $D_\lambda \backslash D$ is closed in D_λ, as D is locally compact and thus open in βD. By the Tietze–Urysohn

Extension Theorem, f is extendible to a continuous bounded function over D_λ, and therefore over $\beta D_\lambda = \beta D$; note that $D \subseteq D_\lambda$. But this is not possible.)

The Tarski Theorem states that for any infinite set I, there is a collection $\mathscr{I}$ of cardinality $|I|^{\aleph_0}$ consisting of countable infinite subsets of I, such that the intersection of any two distinct elements of $\mathscr{I}$ is finite. (See [5].) Note that the collection of all subsets of cardinality at most $\mathfrak{m}$ in a set of cardinality $\mathfrak{n} \geq \mathfrak{m}$ has cardinality at most $\mathfrak{n}^{\mathfrak{m}}$.

A space X is called *linearly Lindelöf* if every linearly ordered (by $\subseteq$) open cover of X has a countable subcover, equivalently, if every uncountable subset of X has a complete accumulation point in X. (A point $x \in X$ is called a *complete accumulation point* of a set $A \subseteq X$ if $|U \cap A| = |A|$ for any open neighborhood U of x in X.)

Observe that if X is a space and $D \subseteq X$, then

$$U \cap \mathrm{cl}_X D = \mathrm{cl}_X(U \cap D)$$

for any open-closed subspace U of X.

Theorem 3.4. *Let X be a locally separable metrizable space. Then $C_S(X)$ is a Banach algebra isometrically isomorphic to $C_\infty(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely $Y = \lambda_{C_S(X)}X$. Moreover*

(1) *Y is countably compact.*
(2) *Y is non-normal, provided that X is non-separable.*
(3) *$C_\infty(Y) = C_K(Y)$.*
(4) *$\dim C_S(X) = d(X)^{\aleph_0}$.*
(5)

$$\begin{aligned} C_S(X) &= \big\{f \in C^*(X) : \mathrm{supp}(f) \textit{ is Lindelöf}\big\} \\ &= \big\{f \in C^*(X) : \mathrm{supp}(f) \textit{ is linearly Lindelöf}\big\} \\ &= \big\{f \in C^*(X) : \mathrm{supp}(f) \textit{ is second countable}\big\}. \end{aligned}$$

Proof. Let $\mathscr{P}$ be separability and $\mathscr{Q}$ be metrizability. Then the pair $\mathscr{P}$ and $\mathscr{Q}$ satisfy the assumption of Theorem 3.1. (Observe that any subspace of a separable metrizable space is separable.) As remarked in Part 1.3 the space X may be represented as a disjoint union

$$X = \bigcup_{i \in I} X_i,$$

where I is an index set and X_i is a non-empty separable open-closed subspace of X for each $i \in I$. To simplify the notation, for any $J \subseteq I$ denote

$$H_J = \bigcup_{i \in J} X_i.$$

Observe that H_J is open-closed in X, thus it has open-closed closure in βX. Also

$$\lambda_{C_S(X)}X = \bigcup\{\mathrm{cl}_{\beta X} H_J : J \subseteq I \text{ is countable}\}.$$

To show this, let $C \in \mathrm{Coz}(X)$ be separable. Then C is Lindelöf and therefore $C \subseteq H_J$ for some countable $J \subseteq I$. Thus $\mathrm{cl}_{\beta X} C \subseteq \mathrm{cl}_{\beta X} H_J$. On the other hand, if $J \subseteq I$ is countable, then H_J is a cozero-set in X, as it is open-closed in X, and it is separable. Since $\mathrm{cl}_{\beta X} H_J$ is open in βX we have

$$\mathrm{cl}_{\beta X} H_J = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} H_J \subseteq \lambda_{C_S(X)}X.$$

Note that (3) implies (1). (3). Let $A=\bigcup_{n\geq 1}A_n$, where each A_n is compact, be a σ-compact subspace of $\lambda_{C_S(X)}X$. By compactness and the definition of $\lambda_{C_S(X)}X$ we have

$$A_n\subseteq \mathrm{cl}_{\beta X}H_{J_1}\cup\cdots\cup \mathrm{cl}_{\beta X}H_{J_{k_n}} \tag{3.1}$$

for some countable $J_1,\ldots,J_{k_n}\subseteq I$. Let

$$J=\bigcup_{n=1}^{\infty}(J_{k_1}\cup\cdots\cup J_{k_n}). \tag{3.2}$$

Then J is countable and $A\subseteq \mathrm{cl}_{\beta X}H_J$. That is A is contained in the compact subspace $\mathrm{cl}_{\beta X}H_J$ of $\lambda_{C_S(X)}X$.

(2). Let $x_i\in X_i$ for each $i\in I$. Then $D=\{x_i:i\in I\}$ is a closed discrete subspace of X, and since X is non-separable, it is uncountable. Suppose to the contrary that $\lambda_{C_S(X)}X$ is normal. Then

$$\lambda_{C_S(X)}X\cap \mathrm{cl}_{\beta X}D=\bigcup\{\mathrm{cl}_{\beta X}H_J\cap \mathrm{cl}_{\beta X}D:J\subseteq I\text{ is countable}\}$$

is normal, as it is closed in $\lambda_{C_S(X)}X$. Now, let $J\subseteq I$ be countable. Since $\mathrm{cl}_{\beta X}H_J$ is open-closed in βX we have

$$\mathrm{cl}_{\beta X}H_J\cap \mathrm{cl}_{\beta X}D=\mathrm{cl}_{\beta X}(\mathrm{cl}_{\beta X}H_J\cap D)=\mathrm{cl}_{\beta X}(H_J\cap D)=\mathrm{cl}_{\beta X}\big(\{x_i:i\in J\}\big).$$

But $\mathrm{cl}_{\beta X}D=\beta D$, as D is closed in (the normal space) X. Therefore

$$\mathrm{cl}_{\beta X}\big(\{x_i:i\in J\}\big)=\mathrm{cl}_{\beta X}\big(\{x_i:i\in J\}\big)\cap \mathrm{cl}_{\beta X}D=\mathrm{cl}_{\beta D}\big(\{x_i:i\in J\}\big).$$

Thus

$$\lambda_{C_S(X)}X\cap \mathrm{cl}_{\beta X}D=\bigcup\{\mathrm{cl}_{\beta D}E:E\subseteq D\text{ is countable}\}=D_\lambda,$$

contradicting the fact that D_λ is not normal.

(4). Since X is non-separable, I is infinite and $d(X)=|I|$. Let $\mathscr{I}$ be a collection of cardinality $|I|^{\aleph_0}$ consisting of countable infinite subsets of I, such that the intersection of any two distinct elements of $\mathscr{I}$ is finite. Let $f_J=\chi_{H_J}$ for any $J\in\mathscr{I}$. No element in

$$\mathscr{F}=\{f_J:J\in\mathscr{I}\}$$

is a linear combination of other elements (as each element of $\mathscr{I}$ is infinite and each pair of distinct elements of $\mathscr{I}$ has finite intersection). Observe that $\mathscr{F}$ is of cardinality $|\mathscr{I}|$. Thus

$$\dim C_S(X)\geq|\mathscr{I}|=|I|^{\aleph_0}=d(X)^{\aleph_0}.$$

On the other hand, if $f\in C_S(X)$, then $\mathrm{supp}(f)$ is Lindelöf (as it is separable) and thus $\mathrm{supp}(f)\subseteq H_J$, where $J\subseteq I$ is countable; therefore, we may assume that $f\in C^*(H_J)$. Conversely, if $J\subseteq I$ is countable, then each element of $C^*(H_J)$ can be extended trivially to an element of $C_S(X)$ (by defining it to be identically 0 elsewhere). Thus $C_S(X)$ may be viewed as the union of all $C^*(H_J)$, where J runs over all countable subsets of I. Note that if $J\subseteq I$ is countable, then H_J is separable; thus any element of $C^*(H_J)$ is determined by its value on a countable set. This implies that for each countable $J\subseteq I$, the set $C^*(H_J)$ is of cardinality at most $\mathfrak{c}^{\aleph_0}=2^{\aleph_0}$. There are at most $|I|^{\aleph_0}$ countable $J\subseteq I$. Therefore

$$\begin{aligned}\dim C_S(X)\leq\big|C_S(X)\big| &\leq \Big|\bigcup\big\{C^*(H_J):J\subseteq I\text{ is countable}\big\}\Big|\\ &\leq 2^{\aleph_0}\cdot|I|^{\aleph_0}=|I|^{\aleph_0}=d(X)^{\aleph_0}.\end{aligned}$$

(5). A metrizable space is Lindelöf if and only if it is second countable, and being Lindelöf implies being linearly Lindelöf. But any linearly Lindelöf locally separable metrizable space T is necessarily Lindelöf; to see this, assume a representation of T as a disjoint union

$$T = \bigcup_{i \in J} T_i,$$

where J is an index set and T_i is a non-empty separable open-closed subspace of T for each $i \in J$. Choose some $t_i \in T_i$ for each $i \in J$ and let $E = \{t_i : i \in J\}$. Suppose to the contrary that J is uncountable. Then E is uncountable and therefore E has a complete accumulation point, say s. If $i \in J$ is such that $s \in T_i$, then T_i is an open neighborhood of s in T intersecting E only in t_i. This contradiction shows that J is countable. Therefore T is separable and thus Lindelöf. To complete the proof, observe that if $\mathrm{supp}(f)$ is linearly Lindelöf for some $f \in C^*(X)$, since $\mathrm{supp}(f)$ is a locally separable metrizable space, $\mathrm{supp}(f)$ is Lindelöf. □

Theorem 3.5. *Let X be a locally Lindelöf paracompact space. Then $C_L(X)$ is a Banach algebra isometrically isomorphic to $C_\infty(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y, namely $Y = \lambda_{C_L(X)}X$. If X is moreover locally compact then*

(1) *Y is countably compact.*
(2) *Y is non-normal, provided that X is non-Lindelöf.*
(3) *$C_\infty(Y) = C_K(Y)$.*
(4) *$\dim C_L(X) = \ell(X)^{\aleph_0}$.*
(5)

$$\begin{aligned} C_L(X) &= \big\{f \in C^*(X) : \mathrm{supp}(f) \textit{ is } \sigma\textit{-compact}\big\} \\ &= \big\{f \in C^*(X) : \mathrm{supp}(f) \textit{ is linearly Lindelöf}\big\}. \end{aligned}$$

Proof. Let $\mathscr{P}$ be the Lindelöf property and $\mathscr{Q}$ be paracompactness. We need to know that the pair $\mathscr{P}$ and $\mathscr{Q}$ satisfy the assumption of Theorem 3.1. It is well known that $\mathscr{Q}$ is hereditary with respect to closed subspaces. (See Corollary 5.1.29 of [3].) It is obvious that $\mathscr{P}$ and $\mathscr{Q}$ satisfy conditions (1), (2) and (4) of Theorem 3.1. It is also known that any paracompact space with a dense Lindelöf subspace is Lindelöf (see Theorem 5.1.25 of [3]), that is, $\mathscr{P}$ and $\mathscr{Q}$ satisfy conditions (3) of Theorem 3.1. The first part of the theorem now follows from Theorem 3.1. The proofs for (1)–(4) are analogous to the ones we have given for the corresponding parts of Theorem 3.4. (One needs to assume a representation for X as given in Part 1.4.) To prove (5), observe that σ-compactness and the Lindelöf property coincide in the realm of locally compact spaces. (See Problem 3.8.C of [3].) Also, observe that local compactness is hereditary with respect to closed subspaces. Thus, for any $f \in C^*(X)$, since $\mathrm{supp}(f)$ is locally compact (as it is closed in X), $\mathrm{supp}(f)$ is σ-compact if and only if it is Lindelöf. Finally, an argument analogous to the one we have given in the proof of Theorem 3.4 (5) shows that any linearly Lindelöf locally compact paracompact space is Lindelöf. Therefore, if $f \in C^*(X)$ and $\mathrm{supp}(f)$ is linearly Lindelöf, since $\mathrm{supp}(f)$ is a locally compact paracompact space, $\mathrm{supp}(f)$ is Lindelöf. □

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